Image Recovery via Multiscale Total Variation

Leonid Rudin [*]
Vicent Caselles [†]

November 21, 1998

Abstract

Total Variation (TV) methods, introduced by L. Rudin and S. Osher a few years ago, are very effective for the recovery of blocky, possibly discontinuous images, from noisy data. Unfortunately they are not so effective for the recovery of textured regions. To improve this, recently, L. Rudin introduced a novel functional, the Multiscale Total Variation (MTV). The functional is designed so that it takes into account the interaction between scales. In this paper we shall present both theoretical and experimental justification of the method. We study the MTV functional and give some existence and uniqueness results for the PDE associated to the functional via the steepest descent method. Then we end by displaying some numerical experiments that show the effectiveness of the method.

Key words: Total variation, bounded variation, denoising, image reconstruction, variational problems, ill-posed problems, constrained partial differential equations.

AMS (MOS) subject classification: 68U10, 49J10, 49J27, 65M05, 65M10

Acknowledgement: The second author was partially supported by EC project MMIP, reference ERBCHRXCT930095 and DGICYT project, reference PB94-1174.

---

[*] R&D Director, Cognitech, Inc., 2800-28th St., Suite 101, Santa Monica, CA 90405, USA and Officer, Sci. Investigations Dept., LAPD (Res. Corp.) USA, 102627.1664@compuserve.com

[†] Dep. of Mathematics and Informatics, University of Illes Balears, 07071 Palma de Mallorca, Spain, dmivca0@ps.uib.es

1 Introduction

L. Rudin and S. Osher proposed some years ago the following method for image reconstruction (see [18], [21] ,[22], [23]). Suppose your image (or your data) u_0 is a function defined on a bounded and smooth or piecewise smooth open subset Ω of $\mathbb{R}^n$ -tipically, Ω will be rectangle in $\mathbb{R}^2$ - and suppose that this data u_0 is a piecewise smooth image u that has been transformed via a linear operator A, tipically a blur, and to which some random noise n has been added. Thus

$$u_0 = Au + n \tag{1}$$

Our purpose is to recover u from the observed data u_0. We must assume some knowledge of A and n in order to solve the problem. The approach proposed by Rudin and Osher [22] was to solve the following constrained minimization problem

$$\begin{aligned} &\text{Minimize} \int_\Omega |\nabla u| \\ &\text{with} \quad \int_\Omega Au = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega |Au - u_0|^2 = \sigma^2 \end{aligned} \tag{2}$$

The first constraint corresponds to the assumption that the noise has zero-mean and the second that its standard deviation is σ. This problem was also studied via a PDE formulation (corresponding to the steepest descent method) in [18] and, recently, in a variational formulation in [9]. It will be our purpose here to extend their results to the present situation where we wish to minimize the MTV functional, i.e.,

$$\begin{aligned} &\text{Minimize} \int_\Omega \frac{|\nabla u|}{(\eta^2 + |\nabla G_t * u|^2)^{1/2}} + \beta \int_\Omega (Ku - u)^p \\ &\text{with} \quad \int_\Omega Au = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega |Au - u_0|^2 = \sigma^2 \end{aligned} \tag{3}$$

where $t > 0$, $\beta \geq 0$, $p \geq 1$, the meaning of the constraints being the same as above. The first term of the functional is introduced with the purpose of regularizing the boundaries of the level sets of the image u_0 and its discontinuities while at the same time we try to keep more features and oscillations. To achieve this the first term contains an interaction between two scales. Here we took the scale corresponding to $\sigma = 0$ in the numerator and $\sigma = \sqrt{t}$ in the denominator. Our numerical experiments will support our previous assertions. Several scales can be incorporated if, instead of (3), we minimize

$$\sum_{i=1}^{k} \int_\Omega \frac{|\nabla G_{t_i} * u|}{(\eta^2 + |\nabla G_{t_{i+1}} * u|^2)^{1/2}} + \beta \int_\Omega (Ku - u)^p \tag{4}$$

with the same constraints as above. Here $0 \leq t_0 < t_1 < ... < t_k$, $k \geq 1$, $\beta \geq 0$, $p \geq 1$. The analysis of (4) being the same as that of (3) we shall restrict us to the simpler model (3).

Let us explain the plan of the paper. In Section 2 we study of the MTV functional from the variational point of view and prove the existence of minimizers for (3). In Section 3 we study the assymptotic limits of the main term (the first term) of the MTV functional as $t \to 0+$ and $t \to +\infty$ showing its relationship with the total variation. Section 4 is devoted to the study of the PDE obtained from (3) via the steepest descent method. We prove existence and uniqueness results and study its assymptotic behaviour. Finally, Section 5 will be devoted to numerical experiments.

2 Existence of minimizers for the MTV functional

Before going into the precise set of assumptions required to study (3), let us make precise the meaning of the first integral in (3). For that let us recall that a function $u \in L^1(\Omega)$ whose derivatives in the sense of distributions are measures with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. If $u \in BV(\Omega)$, the gradient of u will be a vector measure with finite total variation $\|\nabla u\|$

$$\|\nabla u\| = sup\{\int_\Omega u \, div \, v dx : v \in C_0^\infty(\Omega, I\!R^n), |v(x)| \leq 1, x \in \Omega\}$$

Endowed with the norm

$$\|u\|_{BV} = \|u\|_1 + \|\nabla u\|$$

$BV(\Omega)$ is a Banach space. $BV(\Omega)$ is continuously embedded in $L^r(\Omega)$ for all $r \leq \frac{n}{n-1}$ if $n \geq 2$ and all $r \in [0, +\infty[$ if $n = 1$, the inclusion being compact if $r < \frac{n}{n-1}$ in case $n \geq 2$ and for all $r \in [0, +\infty[$ if $n = 1$. Let us define

$$J_t(u) : L^p(\Omega) \to I\!R_+$$

$$J_t(u) = \begin{cases} \int_\Omega \frac{|\nabla u|}{(\eta^2 + |\nabla G_t * u|^2)^{1/2}} & \text{if } u \in BV(\Omega) \\ \infty & \text{if } u \in L^p(\Omega) \setminus BV(\Omega) \end{cases} \tag{5}$$

where we take $p \leq \frac{n}{n-1}$ if $n \geq 2$ and $p \in [0, +\infty[$ if $n = 1$. Concerning the sense of the integral above, observe that, if $u \in L^p(\Omega)$, $\nabla G_t * u \in C^\infty(\bar{\Omega})$. Then $\varphi_t(x) = (\eta^2 + |\nabla G_t * u|^2)^{-1/2} \in C^\infty(\bar{\Omega})$. Let $u \in BV(\Omega)$. Since $|\nabla u|$ is a regular measure on Ω, writing $\mu_u = |\nabla u|$, the definition of $J_t(u)$ must be understood as

$$J_t(u) = \int_\Omega \varphi_t(x) d\mu_u(x).$$

Obviously, $Dom(J_t) = \{u \in L^p(\Omega) : J_t(u) < \infty\} = BV(\Omega)$. On the other hand, observe that if $u \in L^1(\Omega)$ and $J_t(u) < \infty$ then $u \in BV(\Omega)$. From this it may seem at first sight that both functionals have a similar behavior in $BV(\Omega)$. But there is a basic difference between the total variation and $J_t(u)$. Indeed if A is any linear operator in $L^2(\Omega)$ such that $A1 = 1$ then $|\nabla u|(\Omega) + \|Au\|_2$ is coercive on $BV(\Omega)$ ([9], p.2, [1]). This is not the case any more if we replace the total variation $|\nabla u|(\Omega)$ by $J_t(u)$ as is shown in the next remark.

Remark. Let $\Omega = [-1, 1]$ and let us give an example of a sequence u_k such that $J_t(u_k)$ is bounded, $\int_{-1}^{1} u_k(x)dx = 0$ and $TV(u_k) = \int_{-1}^{1} |u_k{}'|dx \to \infty$. Let a_k be any sequence such that $a_k > 0$ and $a_k \to \infty$. Let $u_k = a_k\chi_{[-\frac{3}{4},-\frac{1}{4}]} - a_k\chi_{[-\frac{3}{4},-\frac{1}{4}]}$. Obviously $\int_{-1}^{1} u_k(x)dx = 0$. A simple computation shows that $J_t(u_k)$ is bounded. On the other hand $TV(u_k) = 4a_k \to \infty$ as $k \to \infty$. Morteover $\|u_k\|_2 = a_k^2 \to \infty$ as $k \to \infty$. Hence we cannot guarantee an L^p bound fou u_k if $J_t(u_k)$ is bounded even if $\int_\Omega u_k$ is bounded. But, if $u_k \geq 0$ and $\int_\Omega u_k$ is bounded, then $\nabla G_t * u_k$ is bounded in $L^\infty(\Omega)$ and, being $J_t(u_k)$ bounded we conclude that u_k is bounded in $BV(\Omega)$. Thus additional assumptions will be required to guarantee an L^1 bound for minimizing sequences.

Let us summarize our set of assumptions:

$H1$. $u_0 \in L^2(\Omega)$,
$H2$. A is a continuous linear operator on $L^p(\Omega)$,
$H3$. $A1 = 1$ and $A^*1 = 1$,
$H4$. $n(x)$ is an oscillatory function representing white noise added to the 'clean' image. Thus, we know that $\int_\Omega n = 0$ and $\int_\Omega |n|^2 = \sigma^2$,
$H5$. K is a continuous linear map on $L^p(\Omega)$ such that $K1 = 1$, $K^*1 = 1$ and $I - K$ is invertible on $L^p(\Omega) \cup \{u : \int_\Omega u = 0\}$.

Here we take $p = 2$ if $n = 1, 2$ and $p = \frac{n}{n-1}$ if $n \geq 3$. In some cases it can be convenient to take $p \in [\frac{n}{n-1}, 2]$. For instance, if A is the identity operator, we may stay in $L^2(\Omega)$ even if $n \geq 3$.

To guarantee an L^1 bound for minimizing sequences we may use two different approaches. If $u_0 \geq 0$, we take $\beta = 0$ and minimize $F(u) = J_t(u)$ on the admissible class of functions $\mathcal{A} = \{u \in BV(\Omega) : u \geq 0\}$ which satisfy the constraints. In other case, we need to consider the term $\beta \int_\Omega (Ku - u)^p$ with $\beta > 0$ and we minimize

$$F(u) = J_t(u) + \beta \int_\Omega (Ku - u)^p$$

on the admissible clas of functions $\mathcal{A} = BV(\Omega)$ which satisfy the constraints.

Without loss of generality, we may asume that $\lambda(\Omega) = 1$, where λ denotes the Lebesgue measure on $I\!R^n$. Using that $\int_\Omega Au \cdot n = \int_\Omega n = 0$, a simple computation shows that

$$\|u_0 - c\|_2^2 \geq \sigma^2 \quad \text{for all } c \in I\!R,$$

In particular

$$\|u_0 - \int_\Omega u_0\|_2 \geq \sigma$$

Now we may state our existence result for the problem

$$\begin{array}{l} \text{Minimize} \quad F(u) \\ \ u \in \mathcal{A} \\ \text{with} \quad \displaystyle\int_\Omega Au = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega |Au - u_0|^2 = \sigma^2 \end{array} \tag{6}$$

Theorem 1 . *Assume that H1-H5 above hold. Let* $\sigma \in]0, \|u_0 - \int_\Omega u_0\|_2[$. *Let* $u_0 \in X$ *where* X *is the closure in* $L^2(\Omega)$ *of* $L^2(\Omega) \cap A(L^p(\Omega) \cap BV(\Omega))$. *Then (6) has a solution* $u \in \mathcal{A}$.

Proof. Let $u_0 \in X$. Let $\bar{u}_0 = u_0 - \int_\Omega u_0$. If $u_0 \geq 0$ change the admissible set $\mathcal{A}$ into $\mathcal{A}' = \mathcal{A} - \int_\Omega u_0$. In other case, we may write $\mathcal{A}' = \mathcal{A}$. Let $\bar{u}$ be a solution of (6) with $\mathcal{A}$ replaced by $\mathcal{A}'$ corresponding to $\bar{u}_0$. Observe that $\int_\Omega \bar{u} = \int_\Omega A\bar{u} = \int_\Omega \bar{u}_0 = 0$. Let $u = \bar{u} + \int_\Omega u_0$. Then

$$\int_\Omega Au = \int_\Omega u = \int_\Omega u_0 = 0$$

$$\int_\Omega (Au - u_0)^2 = \int_\Omega (A\bar{u} - \bar{u}_0)^2 = \sigma^2$$

Then if

$$F(\bar{u}) \leq F(v) \quad \text{for all } v \in \mathcal{A}'$$

such that

$$\int_\Omega v = 0 \quad \text{and} \quad \int_\Omega (Av - \bar{u}_0)^2 = \sigma^2$$

it follows that

$$F(\bar{u}) \leq F(v) \quad \text{for all } v \in \mathcal{A}$$

such that

$$\int_\Omega v = \int_\Omega u_0 \quad \text{and} \quad \int_\Omega (Av - u_0)^2 = \sigma^2$$

Thus, instead of u_0 we take $\bar{u}_0$ as data. We have $\int_\Omega \bar{u}_0 = 0$ and our functions will stay in $\{u \in L^p(\Omega) : \int_\Omega u = 0\}$ which is a closed subspace of $L^p(\Omega)$.

Now, let u_k be a minimizing sequence for J in $\mathcal{A}'$ satisfying the constraints. Since $J(u_k)$ is bounded, if $\beta > 0$, we have

$$\int_\Omega (Ku_k - u_k)^p \leq \frac{1}{\beta} M$$

for some constant $M > 0$. On the other hand, by $H5$, $I - K$ is an invertible map from $L^p(\Omega) \cap \{u : \int_\Omega u = 0\}$ onto $Im(I - K)$. Since $\int_\Omega u_k = 0$, $u_k = (I - K)^{-1}(I - K)u_k$ is bounded in $L^p(\Omega)$. If $u_0 \geq 0$, then $\bar{u}_0 \geq -c$ (where $c = \int_\Omega u_0$) and we assume that $\beta = 0$. Then $\int_\Omega |u_k + c| = c$ and it follows that u_k is bounded in $L^1(\Omega)$. In any case $|\nabla G_t * u_k|$ is bounded in $L^\infty(\Omega)$ and since $J_t(u_k)$ is bounded it follows that u_k

is bounded in $BV(\Omega)$. Then, passing to a subsequence if necessary, we may assume that

$$\begin{array}{ll} u_k \to u & \text{weakly in } L^p(\Omega) \\ \nabla u_k \to \nabla u & \text{weakly as measures in } \Omega \end{array} \tag{7}$$

Moreover, Au_k is bounded in $L^2(\Omega)$ and we may assume that $Au_k \to \phi$ weakly in $L^2(\Omega)$. Since A is a continuous linear map in $L^p(\Omega)$, we have $Au_k \to Au$ weakly in $L^p(\Omega)$. Then $\phi = Au$. Since $\nabla G_t * u_k \to \nabla G_t * u$ in $C(\bar{\Omega})$, we see that

$$\int_\Omega \frac{|\nabla u|}{(\eta^2 + |\nabla G_t * u|^2)^{1/2}} \leq \liminf_{k\to\infty} \int_\Omega \frac{|\nabla u_k|}{(\eta^2 + |\nabla G_t * u_k|^2)^{1/2}}$$

We also have

$$\beta \int_\Omega (Ku - u)^p \leq \liminf_{k\to\infty} \beta \int_\Omega (Ku_k - u_k)^p$$

Hence

$$F(u) \leq \liminf_{k\to\infty} F(u_k)$$

and also

$$\int_\Omega Au = 0$$

$$\|Au - \bar{u}_0\|_2 \leq \liminf_{k\to\infty} \|Au_k - \bar{u}_0\|_2 = \sigma$$

As in [9] we define $f(s) = \|sAu - \bar{u}_0\|_2$. $f(s)$ is a continuous function in $[0,1]$ such that $f(0) \geq \sigma$ and $f(1) \leq \sigma$. Then there exists some $s \in [0,1]$ such that $f(s) = \sigma$. Function $u' = su$ satisfies $\int_\Omega u' = 0$, $\|Au' - \bar{u}_0\|_2 = \sigma$. If $s = 0$, we have $\|\bar{u}_0\|_2 = \sigma$. In that case we may take $u = 0$ as the solution of our problem. Now it is easy to check that if $s \in]0,1[$ then $F(u') < F(u)$ unless $\nabla u = 0$. This contradicts the fact that $F(u)$ is the infimum of our problem. If $\nabla u = 0$ we get that $u = 0$ is a solution and $\|\bar{u}_0\|_2 = \sigma$. Hence we may assume that $s = 1$ and $\|Au - \bar{u}_0\|_2 = \sigma$. We conclude that u satisfies the constraints. □

Remarks. 1) If A is invertible we may take $\beta = 0$. In that case, from $\|Au_k - \bar{u}_0\|_2 = \sigma$ it follows that u_k is bounded in $L^2(\Omega)$.
2) In case that $a \leq u_0 \leq b$ we may take as admissible set $\mathcal{A} = BV(\Omega) \cap \{u : a \leq u \leq b\}$ and the constant β can be also taken negative.
3) If the operator K acts on $L^2(\Omega)$ and satisfies $H5$ on this space we may take $\mathcal{A} = BV(\Omega) \cap L^2(\Omega)$ and get the same results as above.

Our next purpose is to show that if u minimizes (6) then u will be a critical point of some related functional. For that we first check that the following problems are equivalent

$$\begin{array}{ll} \text{Minimize} & F(u) \\ u \in \mathcal{A} & \\ \text{with} & \int_\Omega Au = \int_\Omega u_0 \quad \text{and} \quad \|Au - u_0\|_2 = \sigma \end{array} \tag{8}$$

$$\begin{array}{ll} \text{Minimize} & F(u) \\ u \in \mathcal{A} & \\ \text{with} & \int_\Omega Au = \int_\Omega u_0 \quad \text{and} \quad \|Au - u_0\|_2 \leq \sigma \end{array} \tag{9}$$

To check the equivalence between (8) and (9) we observe that if $v \in \mathcal{A}$, $\|Av - u_0\|_2 \leq \sigma$ attains the infimum of (9), defining $f(s) = \|sAv - u_0\|_2$ and arguing as above we show that $\|Av - u_0\|_2 = \sigma$. It follows that (8) and (9) are equivalent.

To state our next result we shall need some notation. Obviously the functional $J_t(u)$ is not convex. Let

$$J_t(.,.) : L^p(\Omega) \times L^p(\Omega) \to \mathbb{R}_+$$

be defined by

$$J_t(u,v) = \int_\Omega \frac{|\nabla u|}{(\eta^2 + |\nabla G_t * v|^2)^{1/2}} \tag{10}$$

and let

$$F(u,v) = J_t(u,v) + \beta \int_\Omega (Ku - u)^p \tag{11}$$

Then for all $v \in L^p(\Omega)$ fixed the functional $u \to F(u,v)$ is a convex functional. Let us write $\partial_1 F(u,v)$ to refer to the subdifferential of this functional at u defined on the set $u_0 + \{v \in L^p(\Omega) : \int_\Omega v = 0\}$. Recall that given a Banach space E, $u_0 \in E$ and a convex, proper functional $\phi : u_0 + E \to]-\infty, +\infty]$, the subdifferential of ϕ at $u \in u_0 + E$, $\partial\phi(u)$, is given by

$$\partial\phi(u) = \{w \in E' : \phi(v) - \phi(u) \geq < w, v - u > \forall v \in u_0 + E\}$$

Notice that it coincides with the subdifferential of the same functional taken in $L^p(\Omega)$. On the other hand, for any $u \in BV(\Omega)$ fixed, the functional $v \to F(u,v)$ is Gateaux differentiable at any $v \in u_0 + \{v' \in L^p(\Omega) : \int_\Omega v' = 0\}$. Let us denote this derivative by $\partial_2 F(u,v)$. It is easy to see that it coincides with the Gateaux derivative taken in $L^p(\Omega)$ plus some constant. Now we can state

Proposition 1 . *Let u be a solution of (8) and assume that $A : L^p(\Omega) \to L^2(\Omega)$ is continuous. Then there exists some $\lambda_0 \geq 0$ such that*

$$-\lambda_0 A^*(Au - u_0) \in \partial_1 F(u,u) + \partial_2 F(u,u) \tag{12}$$

Proof. Let us redefine F so that

$$F(u) = \begin{cases} F(u) & \text{if } u \in u_0 + \{v \in L^p(\Omega) : \int_\Omega v = 0\} \\ \infty & \text{if } u \notin u_0 + \{v \in L^p(\Omega) : \int_\Omega v = 0\} \end{cases}$$

Write

$$G(u) = \chi_{u_0+\sigma\bar{B}}(u) = \begin{cases} 0 & \text{if } u \notin u_0 + \sigma\bar{B} \\ \infty & \text{if } u \in u_0 + \sigma\bar{B} \end{cases}$$

where $\bar{B}$ denotes the closed unit ball of $L^2(\Omega)$. G is a convex functional. Observe that F and G are both lower semicontinuous on $L^p(\Omega)$, $L^2(\Omega)$, respectively . Then using this notation (9) is equivalent to

$$\begin{array}{ll} \text{Minimize} & F(u) + G(Au) \\ u \in L^p(\Omega) & \end{array} \tag{13}$$

Observe that $Dom\ F = BV(\Omega) \cap \{v \in L^p(\Omega) \int_\Omega v = \int_\Omega u_0\}$, $DomG = u_0 + \sigma\bar{B}$. Since u is a solution of (8), hence of (9), we have

$$F(u) + G(Au) \leq F(v) + G(Av) \qquad \text{for all } v \in L^p(\Omega)$$

For convenience, let us write $F(u,u)$ instead of $F(u)$. then

$$\begin{aligned} F(u,u) &+ G(Au) \leq F((1-s)u + sv, (1-s)u + sv) + \\ &+ G(A((1-s)u + sv)) \leq (1-s)F(u, (1-s)u + sv) + \\ &+ sF(v, (1-s)u + sv) + (1-s)G(Au) + sG(Av) \end{aligned}$$

for any $s \in [0,1]$.Observe that $G(Au) < \infty$ and if $G(Av) < \infty$ then also $G(A((1-s)u+sv)) < \infty$. Then, taking into account that $G(Au) < \infty$ and $F(u, (1-s)u+sv) < \infty$ the previous expression may be written as

$$\begin{aligned} F(u,u) &- F(u, (1-s)u + sv) \leq sF(v, (1-s)u + sv) - \\ &- sF(u, (1-s)u + sv) + sG(Av) - sG(Au) \end{aligned}$$

Dividing the last expression by s and letting $s \to 0+$ we get that

$$- < \partial_2 F(u,u), v - u > \leq F(v,u) - F(u,u) + G(Av) - G(Au)$$

In other words

$$-\partial_2 F(u,u) \in \partial(F(.,u) + G(A.))(u)$$

Now recall that we assumed $u_0 \in \overline{ADomF}$. Then there exists $\tilde{u} \in DomF$ with $\|A\tilde{u} - u_0\|_2 < \frac{\sigma}{2}$. Since A is continuous from $L^p(\Omega)$ into $L^2(\Omega)$, $G \circ A$ is continuous at $\tilde{u}$ ($\in IntDom(G \circ A)$) and therefore ([11], Prop. 5.6)

$$\partial(F(.,u) + G(A.)) = \partial F(.,u) + \partial G(A.)$$

Moreover, as G is continuous at $A\tilde{u}$ we have

$$\partial(G \circ A)(v) = A^* \partial G(Av) \quad \text{for all}\, v \in L^p(\Omega)$$

([11], Prop. 5.7) where $\partial G(z) = \{0\}$ if $\|z - u_0\|_2 < \sigma$, $\partial G(z) = \{s(z - u_0) : s \geq 0\}$ if $\|z - u_0\|_2 = \sigma$. As any solution u of (8) verifies $\|u - u_0\|_2 = \sigma$, we conclude that there exits some $\lambda_0 \geq 0$ such that

$$-\partial_2 F(u,u) \in \partial_1 F(u,u) + \lambda_0 A^*(Au - u_0)$$

which gives (12). □

In contrast with the situation in [9], since F is not convex, we cannot say that critical points of (12) are minimizers of the corresponding associated functional.

3 Limits of the MTV functional

To understand the meaning of the functional J_t given by (5) we study its assymptotic behaviour as $t \to 0+$, $t \to +\infty$, respectively. We prove the following theorem.

Theorem 2 . *Let $u \in BV(\mathbb{R}^n)$. Let $\nabla u = Du + \mu$ be the Lebesgue decomposition of ∇u so that Du denotes the density of ∇u with respect to the Lebesgue measure in $\mathbb{R}^n$. Let*

$$J_t(u) = \int_{\mathbb{R}^n} \frac{|\nabla u|}{(\eta^2 + |\nabla G_t * u|^2)^{1/2}} dx \tag{14}$$

Let

$$J_0(u) = \int_{\mathbb{R}^n} \frac{|Du|}{(\eta^2 + |Du|^2)^{1/2}} dx \tag{15}$$

Then

$$J_t(u) \to J_0(u) \quad \text{as } t \to 0+ \tag{16}$$

and

$$J_t(u) \to \frac{1}{\eta} TV(u) \quad \text{as } t \to \infty \tag{17}$$

If $u = h\chi_C$ where C is an open bounded subset of $\mathbb{R}^n$ with smooth, say C^2, boundary ∂C, then

$$J_t(u) = 2\sqrt{\pi t}(1 + O(t)) H^{n-1}(\partial C) \quad \text{as } t \to 0+ \tag{18}$$

and

$$J_t(u) = \frac{h}{\eta}(1 + O(\frac{1}{t})) H^{n-1}(\partial C) \quad \text{as } t \to \infty \tag{19}$$

Remarks. The assymptotic limits given in (16), (17) show that J_t connects the total variation with the functional J_0. Thus, the results obtained by minimizing $J_t(u)$ for large t should not be far from those obtained by minimizing the total variation ([21],[22],[23]). Formulas (18), (19) show the assymptotic limit for the spacial case where $u = h\chi_C$. Observe that the size of the jump appears explicitely in (19) but not in (18). Thus we may expect that when minimizing J_t for t small we shall regularize the boundaries of the level sets of u without destroying the discontinuities, i.e., with t small we shall keep better the size of the jumps. Accordding to (16), as $t \to 0+$ the functional becomes independent of the discontinuities of u. To give a geometric interpretation of it, let us consider a piecewise smooth image u with jump discontinuities. The normal to the graph of u is given by $N(u) = \frac{1}{(1+|\nabla u|^2)^{1/2}}(-\nabla u, 1)$ a.e. (except on the discontinuity set of u). The functional $J_0(u)$ with $\eta = 1$ is the integral of the horizontal (or (x, y)) component of $N(u)$. Thus, minimizing $J_0(u)$ amounts to minimize the horizontal component of $N(u)$. Intuitively, we can expect that the minimum is attained for a piecewise constant function. Minimizing $J_0(u)$ with the steepest descent method should give a piecewise constant segmentation of the original image. This seems to be the case from the experiments ([7]). But $J_0(u)$ is a nonconvex functional and the steepest descent method leads to a Perona-Malik type equation and not much can be said about it besides it is ill-posed. Using $J_t(u)$ for some $t > 0$ avoids these problems. We are able to regularize the boundaries of the level sets of u and its discontinuities and at the same time we keep more features and oscillations.

Proof. In what follows we denote by λ the Lebesgue measure on $I\!R^n$. The first statement (16) will be a consequence of the following two facts

$$\mu\{x \in I\!R^n : \liminf_{t\to 0+} G_t * \mu(x) < \infty\} = 0 \tag{20}$$

$$\int_{I\!R^n} \varphi_t(x)|Du(x)|dx \to \int_{I\!R^n} \varphi(x)|Du(x)|dx \quad \text{as } t \to 0+ \tag{21}$$

where $\varphi_t(x) = (\eta^2 + |\nabla G_t * u(x)|^2)^{-1/2}$, $\varphi(x) = (\eta^2 + |Du(x)|^2)^{-1/2}$. Assume that (20), (21) have been proved and let us prove (16). From (20) we know that

$$\liminf_{t\to 0+} G_t * \mu(x) = \infty \quad \mu \text{ a.e. on } I\!R^n \tag{22}$$

Hence, $\limsup_{t\to 0+} \varphi_t(x) = 0$ μ a.e. on $I\!R^n$. It follows that

$$\int_{I\!R^n} \varphi_t(x)d\mu \to 0 \quad \text{as } t \to 0+ \tag{23}$$

Using this, (21) and

$$\int_{I\!R^n} \varphi_t(x)|\nabla u|dx = \int_{I\!R^n} \varphi_t(x)|Du|dx + \int_{I\!R^n} \varphi_t(x)d\mu \tag{24}$$

(16) follows immediately. Let us now check (20),(21) above. To prove (20), let

$$H(M)=\{x\in supp\,\mu : \liminf_{t\to 0+}\frac{\mu(B(x,t))}{\lambda(B(x,t))}<M\} \tag{25}$$

and prove that $\mu(H(M))=0$. Let $\varepsilon>0$ be fixed. Since μ is a regular Borel measure and $\lambda(supp\,\mu)=0$, there exists an open subset G of $I\!R^n$ such that $G\supseteq H(M)$, $\lambda(G)<\varepsilon$ and $\mu(G)\le\mu(H(M))+\varepsilon$. Consider

$$\mathcal{V}=\{B(x,t) : B(x,t)\subset G \text{ and } \frac{\mu(B(x,t))}{\lambda(B(x,t))}<M\} \tag{26}$$

It follows from the definition of $H(M)$ that $\mathcal{V}$ is a Vitali covering of $H(M)$. Then ([14], [30]) there exists a countable family of balls $\{Q_i : i\in I\!N\}$, two by two disjoint, such that $\mu\left(H(M)\setminus\bigcup_{i=1}^{\infty}Q_i\right)=0$. Since

$$H(M)=\bigcup_{i=1}^{\infty}(H(M)\bigcap Q_i)\bigcup(H(M)\setminus\bigcup_{i=1}^{\infty}Q_i),$$

we have

$$\begin{aligned}\mu(H(M)) &\le \sum_{i=1}^{\infty}\mu(H(M)\cap Q_i)\le\sum_{i=1}^{\infty}\mu(Q_i)\le\\ &\le M\sum_{i=1}^{\infty}\lambda(Q_i)\le M\lambda(G)<M\varepsilon\end{aligned}$$

It follows that $\mu(H(M))=0$ for all $M>0$. Now, let

$$H^*(M)=\{x\in supp\,\mu : \liminf_{t\to 0+}G_t*\mu<M\} \tag{27}$$

Let ω_n be the constant such that $\lambda(B(0,R))=\omega_n R^n$, $R>0$. Since for $|x|\le 2\sqrt{t}$

$$G_t(x)\ge\frac{1}{e(4\pi t)^{n/2}}=\frac{\omega_n}{e\pi^{n/2}\lambda(B(x,t))}$$

we have

$$G_t\ge\frac{\omega_n}{e\pi^{n/2}}\frac{\chi_B(x,t)}{\lambda(B(x,t))}$$

Then, it is easy to check that $H^*(M)\subseteq H(\frac{Me\pi^{n/2}}{\omega_n})$. It follows that $\mu(H^*(M))=0$. Since this is true for all $M>0$ and $\mu(I\!R^n\setminus supp\,\mu)=0$, (20) follows. To prove (21), we are going to prove that for any sequence $t_k\to 0+$, there exists a subsequence t_{k_j} such that

$$\int_{I\!R^n}\varphi_{t_{k_j}}(x)|Du(x)|dx\to\int_{I\!R^n}\varphi(x)|Du(x)|dx \quad \text{as } j\to\infty$$

This implies (21). Thus, let $t_k \to 0+$. Since $G_{t_k} * \mu \to \mu$ weakly* as measures, we know that $\limsup_{k\to\infty} G_{t_k} * \mu(K) \leq \mu(K)$ for any compact subset K of $I\!R^n$. Since $\varphi_t, \varphi \leq 1/\eta$ and $|Du(x)| \in L^1(I\!R^n)$, given $\varepsilon_r > 0$, $\varepsilon_r(1+\frac{1}{\eta}) < \frac{1}{2^r}$, we can find $R_r > 0$ such that

$$\int_{I\!R^n\setminus B(0,R_r)} \varphi_t(x)|Du(x)|dx < \varepsilon_r$$

$$\int_{I\!R^n\setminus B(0,R_r)} \varphi(x)|Du(x)|dx < \varepsilon_r$$

Now, since λ is regular and $\lambda(supp\,\mu) = 0$, we can find a compact subset $P_r \subseteq B(0,R_r)\setminus supp\,\mu$ such that $\lambda(B(0,R_r)\setminus P_r) < \varepsilon_r$, $\int_{I\!R^n\setminus P_r} |Du(x)|dx < \varepsilon_r$. Let us, for simplicity forget about the subindex r in the notation above. Since $P \cap supp\,\mu = \emptyset$, $\limsup_{k\to\infty} G_{t_k} * \mu(P) \leq \mu(P) = 0$. Thus,

$$\int_P G_{t_k} * \mu(x)dx \to 0 \quad \text{as } k \to \infty$$

Thus, there exists a subsequence t_{k_j} of t_k such that $G_{t_{k_j}} * \mu(x) \to 0$ λ a.e. on P. Now

$$\begin{aligned} &| \int_{I\!R^n} \varphi_{t_{k_j}}(x)|Du(x)|dx - \int_{I\!R^n} \varphi(x)|Du(x)|dx \;| \leq \\ \leq\; & 2\varepsilon + |\int_{B(0,R)} (\varphi_{t_{k_j}}(x) - \varphi(x))|Du(x)|dx| \leq \\ \leq\; & 2\varepsilon + \frac{2}{\eta}\int_{B(0,R)\setminus P} |Du(x)|dx + \int_P |\varphi_{t_{k_j}}(x) - \varphi(x)||Du(x)|dx \end{aligned}$$

Since $\varphi_{t_{k_j}}(x) \to \varphi(x)$ λ a.e. on P, $\varphi_{t_{k_j}}, \varphi \leq \frac{1}{\eta}$ we have that

$$\limsup_{t_{k_j}\to 0+} |\int_{I\!R^n} \varphi_{t_{k_j}}(x)|Du(x)|dx - \int_{I\!R^n} \varphi(x)|Du(x)|dx| \leq 2\varepsilon(1+\frac{1}{\eta}) < \frac{1}{2^r}$$

Then we may find $t_r{}'$ such that

$$|\int_{I\!R^n} \varphi_{t_r{}'}(x)|Du(x)|dx - \int_{I\!R^n} \varphi(x)|Du(x)|dx| < \frac{1}{2^r}$$

Since this can be done for each $r \in I\!N$, (21) follows. On the other hand, (17) follows from the fact that $G_t * \nabla u(x) \to 0$ λ a.e. as $t \to \infty$.

The precise estimates given in (18), (19) require a more detailed analysis. We shall sketch it without giving all details. Basically the proof follows the lines of [13]. Differentiating $G_t * \chi_C(x)$ with respect to x_i, $i = 1, 2, ..., n$,

$$\partial_{x_i} G_t * \chi_C(x) = \frac{1}{(4\pi t)^{n/2}} \int_{I\!R^n} \frac{y_i - x_i}{2t} e^{-\frac{|y-x|^2}{4t}} \chi_C(y)dy$$

We are going to estimate the previous expression for all $x \in \partial C$. Without loss of generality we may assume that $x = 0$ and $0 \in \partial C$. Since we are assuming that ∂C is a smooth surface, in particular a C^2 surface, we may assume that the outer unit normal $\nu(0)$ at ∂C at the point $x = 0$ is $\nu(0) = (0, ..., 0, 1)$ and the boundary of ∂C near $x = 0$ is given by a graph

$$x_n = \gamma(x'),\ x' \in \mathbb{R}^{n-1},\ \|x'\| < \delta$$

(where $\|x'\| = \max\{|x_i'| : i = 1, ..., n-1\}$) for some $\delta > 0$. Notice that $\gamma(0) = 0$, $\nabla\gamma(0) = 0$. Since ∂C is smooth we may choose $\delta > 0$ so that the above construction can be done with the same δ for any point in ∂C. Moreover, if we make the change of variables $y = \delta z$, then we may write

$$\partial_{x_i} G_t * \chi_C(0) = \frac{\delta^{n+1}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \frac{z_i}{2t} e^{-\frac{\delta^2|z|^2}{4t}} \chi_{\frac{C}{\delta}}(z) dz$$

Observe that $\delta^{-1}C$ is the graph of $\gamma_\delta(z) = \frac{\gamma(\delta z)}{\delta}$ defined on $Q' = \{x' \in \mathbb{R}^{n-1} : \|x'\| \leq 1\}$. Let $Q = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Then, we write

$$\partial_{x_i} G_t * \chi_C(0) = \frac{\delta^{n+1}}{(4\pi t)^{n/2}} \frac{1}{2t} \int_{Q \cap \delta^{-1}C} z_i e^{-\frac{\delta^2|z|^2}{4t}} dz +$$

$$+ \frac{\delta^{n+1}}{(4\pi t)^{n/2}} \frac{1}{2t} \int_{(\mathbb{R}^n \setminus Q) \cap \delta^{-1}C} z_i e^{-\frac{\delta^2|z|^2}{4t}} + \chi_{\frac{C}{\delta}}(z) dz = I_{1,i} + I_{2,i}$$

$i = 1, ..., n$. Let us first estimate $I_{1,n}$. First we write

$$I_{1,n} = \frac{\delta^{n+1}}{(4\pi t)^{n/2}} \frac{1}{2t} \int_{Q'} \int_{-1}^{\gamma_\delta(z')} z_n e^{-\frac{\delta^2 z_n{}^2}{4t}} dz_n e^{-\frac{\delta^2|z'|^2}{4t}} dz'$$

Then we write

$$I_{1,n} = T_1 + T_2 - T_3 - T_4$$

where

$$T_1 = \frac{\delta^{n-1}}{(4\pi t)^{n/2}} \int_{Q'} e^{-\frac{\delta^2}{4t}} e^{-\frac{\delta^2|z'|^2}{4t}} dz' = O(e^{-\frac{\alpha}{t}})$$

$$T_2 = \frac{\delta^{n-1}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus Q'} e^{-\frac{\gamma(\delta z')^2}{4t}} e^{-\frac{\delta^2|z'|^2}{4t}} dz' = O(e^{-\frac{\alpha}{t}})$$

$$T_3 = \frac{\delta^{n-1}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^{n-1}} (e^{-\frac{\gamma(\delta z')^2}{4t}} - 1) e^{-\frac{\delta^2|z'|^2}{4t}} dz' = O(\sqrt{t})$$

$$T_4 = \frac{\delta^{n-1}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^{n-1}} e^{-\frac{\delta^2|z'|^2}{4t}} dz' = \frac{1}{2\sqrt{\pi t}}$$

Thus, $I_{1,n} = -\frac{1}{2\sqrt{\pi t}} + O(\sqrt{t})$. A similar analysis along the lines of [13] leads to the estimate $I_{1,i} = O(\sqrt{t})$, $i = 1, ..., n-1$. To estimate the integral $I_{2,i}$ we observe that taking absolute values of the integrand and replacing the region where we are integrating by $\mathbb{R}^n \setminus Q$ the resulting integral can be easily estimated as a $O(e^{-\frac{\alpha}{t}})$ for some $\alpha > 0$. Collecting all these estimates we see that

$$|\nabla G_t * \chi_C(0)| = \frac{1}{2\sqrt{\pi t}}(1 + O(t))$$

and a similar estimate can be obtained for all $x \in \partial C$. Now,

$$J_t(h\chi_C) = h \int_{\partial C} \frac{dS}{(\eta^2 + h^2 |\nabla G_t * \chi_C(x)|^2)^{1/2}} =$$

$$= \frac{2\sqrt{\pi t} h}{(4\pi t \eta^2 + h^2(1 + O(t)))^{1/2}} H^{n-1}(\partial C) = 2\sqrt{\pi t}(1 + O(t)) H^{n-1}(\partial C)$$

which gives (18) (dS being the area element of ∂C). The asymptotics when $t \to \infty$ is simpler and we shall omit the details. □

4 The PDE Approach

For simplicity, we study first the PDE associated to the unconstrained functional where the results are proved with detail. An approximation to the constrained case will be considered in the remark at the end of this section. Since, in that case, the proof of the main result is similar but slightly more cumbersome than the unconstrained one, we shall omit the details.

Let G denote a gaussian function of variance τ. We define $J : L^2(\Omega) \to \mathbb{R}_+$ by

$$J(u) = \begin{cases} \displaystyle\int_\Omega \frac{|\nabla u|}{(\eta^2 + |\nabla G * u|^2)^{1/2}} & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ \infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega) \end{cases} \tag{28}$$

Similarly, let $J(.,.) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}_+$ be given by

$$J(u,v) = \begin{cases} \displaystyle\int_\Omega \frac{|\nabla u|}{(\eta^2 + |\nabla G * v|^2)^{1/2}} & \text{if } u \in BV(\Omega) \cap L^2(\Omega), v \in L^2(\Omega) \\ \infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega), v \in L^2(\Omega) \end{cases} \tag{29}$$

Our purpose will be to write an evolution equation which starting at u_0 will decrease the functional $J(u)$ along the trajectory. The trajectory will converge to a critical point of the functional. For that we need to compute the Euler-Lagrange associated to $J(u)$. Formally, this amounts to write

$$u_t + \partial_1 J(u,u) + \partial_2 J(u,u) = 0 \tag{30}$$

where the subdifferential $\partial_1 J(u,u)$ and the Gateaux derivative $\partial_2 J(u,u)$ at any $u \in L^2(\Omega)$ are computed with respect to the Banach space $L^2(\Omega)$. Let $u \in BV(\Omega) \cap L^2(\Omega)$ and $v \in L^2(\Omega)$. Formally at least,

$$\partial_1 J(u,v) = -div(\Theta(v)\,\frac{\nabla u}{|\nabla u|})$$

where

$$\Theta(v) = \frac{1}{(\eta^2 + |\nabla G * v|^2)^{1/2}}\,.$$

and u must satisfy Neuman boundary conditions. On the other hand the Gateaux derivative $\partial_2 J(u,v)$ can be computed explicitely. For that, given any vector field $Q \in L^1(\Omega, I\!R^n)$, $Q = (Q_1, ..., Q_n)$ and a finite vector measure $\nu = (\nu_1, ..., \nu_n)$ on Ω let us write

$$Q \tilde{*} \nu = Q_1 * \nu_1 + ... + Q_n * \nu_n\,.$$

Then, it is easy to see that

$$\partial_2 J(u,v) = Q \tilde{*} (\Theta(v)^3\, \nabla G * v\, |\nabla u|) \tag{31}$$

Thus, formally, we write (30) as

$$\begin{aligned} & u_t = div(\Theta(u)\,\frac{\nabla u}{|\nabla u|}) - Q \tilde{*} (\Theta(u)^3\, \nabla G * v\, |\nabla u|) \;\; x \in \Omega,\, t > 0 \\ & \frac{\partial u}{\partial n} = 0 \;\text{ on } \partial\Omega \\ & u(0,x) = u^0(x) \text{ in } \Omega \end{aligned} \tag{32}$$

for some initial condition $u^0(x)$. With no constraints we may take $u^0(x) = u_0(x)$ where $u_0(x)$ is the given image. Before going into the details concerning the existence of solutions of (32) let us recall the following standard notation. Let E be a Banach space, $T > 0$.

1. $C([0,T], E) := \{u : [0,T] \to E \text{ continuous}\}$.
2. $L^r([0,T], E) := \{u : [0,T] \to E : \int_0^T \|u(t)\|^r dt < \infty\}$ with $1 \leq r \leq \infty$.
3. $L^\infty([0,T], E) := \{u : [0,T] \to E : \text{ess sup}_{t \in [0,T]} \|u(t)\| < \infty\}$.
4. $u \in L^r_{loc}([0, +\infty[, E)$ means that $u \in L^r([0,T], E)$ for all $T > 0\}$.
5. $W^{1,r}([0,T], E) := \{u : [0,T] \to E : u, u_t \in L^r([0,T], E)\}$

First let us write (30) in the form of an abstract evolution problem :

$$u_t + \partial_1 J(u,u) + \partial_2 J(u,u) = 0 \tag{33}$$

To give sense to this formulation, let us recall the following definition: let $T > 0$, $E = L^2(\Omega)$, $f \in L^1([0,T], E)$. We call $u \in C([0,T], E)$ a strong solution of

$$u_t + \partial_1 J(u,u) \ni f(t) \tag{34}$$

if u is differentiable a. e. on $]0,T[$, $u(t) \in \mathrm{Dom}\, \partial_1 J(.,u(t))$ a.e. on $]0,T[$ and

$$-u_t + f(t) \in \partial_1 J(u(t),u(t)) \text{ a.e. on}]0,T[\tag{35}$$

In particular, if $u \in W^{1,1}([0,T],E)$, $u(t) \in \mathrm{Dom}\, \partial_1 J(.,u(t))$ and (35) holds a.e. on $]0,T[$ then u is a strong solution of (34). We say that $u \in C([0,T],E) \cap L^\infty([0,T],BV(\Omega))$ is a strong solution of (33) if there exists $w(t) \in L^1([0,T],E)$, $w(t) \in -\partial_2 J(u(t),u(t))$ such that

$$w(t) \in u_t + \partial_1 J(u(t),u(t))$$

Now we can use the machinery of nonlinear semigroups in Banach spaces to prove existence of solutions of (33). We have:

Theorem 3 *For any $u^0 \in BV(\Omega) \cap L^\infty(\Omega)$, there exists a strong solution $u \in C([0,T],L^2(\Omega)) \cap L^\infty([0,T],BV(\Omega))$ of (33) with initial datum u^0 such that $\|u(t)\|_\infty \leq \|u^0\|_\infty$. Moreover, the functional $J(u)$ is a Lyapounov functional for (33), i.e., $\frac{d}{dt}J(u(t)) \leq 0$.*

Proof. The proof is divided in two steps.

1st step. Regularization.

Define for each $\varepsilon > 0$ the following functional

$$J^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \frac{\sqrt{\varepsilon^2 + |\nabla u|^2} - \varepsilon}{(\eta^2 + |\nabla G_t * u|^2)^{1/2}} & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ \infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega) \end{cases} \tag{36}$$

Let $u^0_\varepsilon \in W^{1,\infty}(\Omega)$, $\|u^0_\varepsilon\|_\infty \leq \|u^0\|_\infty$, $u^0_\varepsilon \to u^0$ in $L^2(\Omega)$, $\|\nabla u^0_\varepsilon\|_1 \to TV(u^0)$ as $\varepsilon \to 0$. Observe that this implies that $J^\varepsilon(u^0_\varepsilon) \to J^\varepsilon(u^0)$ ([12], [26]). The corresponding associated problem can be written in abstract form

$$\begin{aligned} & u_t + \partial_1 J^\varepsilon(u,u) + \partial_2 J^\varepsilon(u,u) \ni 0 \\ & u(0) = u^0_\varepsilon \end{aligned} \tag{37}$$

or in classical form in $]0,T[\times\Omega$

$$\begin{aligned} & u_t = div(\Theta(u)\frac{\nabla u}{(\varepsilon^2 + |\nabla u|^2)^{1/2}}) - Q\tilde{*}(\Theta(u)^3\, \nabla G * u\, \varphi_\varepsilon(|\nabla u|)) \\ & \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \\ & u(0,x) = u^0_\varepsilon(x) \text{ in } \Omega \end{aligned} \tag{38}$$

An existence result for this approximated problem follows along the lines of [8] using an L^∞ estimate on ∇u which is proved as in [3]. We shall only sketch the details. We introduce the space

$$W(0,T) := \{w \in L^2([0,T], W^{1,2}(\Omega)) : \frac{dw}{dt} \in L^2([0,T], L^2(\Omega))\}$$

This is a Hilbert space for the graph norm. Following [17], let $H_{loc}^{\frac{\gamma}{2},\gamma}([0,T] \times \Omega)$ ($H_{loc}^{\frac{\gamma}{2},\gamma}(Q')$ where $\bar{Q}' \subseteq]0,T[\times\Omega$) be the space of functions which are locally Hoelder continous in t and x of exponents $\frac{\gamma}{2}$ and γ, respectively (Hoelder continous in t and x in Q' with the same exponents as above). Let W_0 be the set of functions in $W(0,T)$ such that

$$\|w\|_{L^\infty([0,T],L^2(\Omega))} \leq C_1$$

$$\|w\|_{L^\infty([0,T],W^{1,2}(\Omega))} \leq C_2$$

$$\|\frac{dw}{dt}\|_{L^2([0,T],L^2(\Omega))} \leq C_3$$

$$\|\nabla w\|_{H^{\frac{\gamma}{2},\gamma}(Q')} \leq C_4(Q')$$

for any Q' such that $\bar{Q}' \subseteq]0,T[\times\Omega$, where $C_4(Q')$ depends on the distance from Q' to the boundary of $]0,T[\times\Omega$, C_1, C_2, C_3 are some constants to be determined below and some $\gamma \in]0,1[$ also to be precised below. Observe that W_0 is a non empty, closed and weakly compact subset of $W(0,T)$. Let $w \in W_0$. Consider the following problem on $]0,T[\times\Omega$

$$\begin{aligned} & u_t = div(\Theta(w)\,\frac{\nabla u}{(\varepsilon^2 + |\nabla u|^2)^{1/2}}) - Q\tilde{*}(\Theta(w)^3\,\nabla G * w\;\varphi_\varepsilon(|\nabla w|)) \\ & \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \\ & u(0,x) = u_\varepsilon^0(x) \text{ in } \Omega \end{aligned} \tag{39}$$

Let

$$B_w(t,x) = Q\tilde{*}(\Theta(w)^3\,\nabla G * w\;\varphi_\varepsilon(|\nabla w|))$$

Then, this problem can be written in abstract form as

$$u_t + \partial_1 J(u,w) \ni -B_w(t,x) \tag{40}$$

A classical result in the theory of nonlinear semigroups ([4], [6]) gives us the existence of a unique strong solution of it satisfying

$$\|u_t\|_{L^2([0,T],L^2(\Omega))} \leq \|B_w\|_{L^2([0,T]\times\Omega)} + (J^\varepsilon(u^0, w(0)))^{1/2} \tag{41}$$

Moreover, the solution of the problems obtained by adding a term $-\rho\Delta u$ at the left hand side converges to the strong solution of (40). Thus the following estimates

$$\|u(t)\|_r \leq \|u_0\|_r + Ct \text{ for all } r \in [1, +\infty[\tag{42}$$

$$\|\nabla u(t)\|_\infty \leq \|\nabla u^0\|_\infty + C' t e^{C'' t} \tag{43}$$

where C, C', C'' are constants depending only on η, $\|\nabla G\|_2$, $\|D^2 G\|_2$, $\|D^3 G\|_2$ and the constants C_1, C_2 in in the definition of W_0 (indeed, they do not depend directly on C_2 but on $TV(w)$), are easily proved for the problems approximating (40) and, passing to the limit, they are also true for (40). Let us only mention that the proof of (43) follows the lines of the analogous estimate given in [3]. Moreover we also see that $u(t,x)$ is also a weak solution, i.e. in the sense of distributions, of (39). On the other hand, let us observe that if $u \in W^{1,r}(\Omega)$ for some $r \geq 1$ and $v \in L^p(\Omega)$, the $\partial_1 J(u,v)$ is given by

$$\partial_1 J(u,v)(u') = \int_\Omega \Theta(v) \frac{\nabla u \cdot \nabla u'}{(\varepsilon^2 + |\nabla u|^2)^{1/2}}$$

Since $u_t \in L^2([0,T]\times\Omega)$ and u is a weak solution of (39), then we see that the divergence term in (39) is also in $L^2([0,T]\times\Omega)$ and the equation holds a.e. in $[0,T]\times\Omega$.

Now, bounded weak solutions of (39) with ∇u bounded satisfy further a priori estimates. First observe that thanks to the L^∞ bound on ∇u we may transform our problem so that the term $div(\Theta(w)\frac{\nabla u}{(\varepsilon^2+|\nabla u|^2)^{1/2}})$ can be written in the form $div(\Theta(w)\, a(\nabla u))$ for some vector $a = (a_1, ..., a_n)$ such that

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial p_j}(p)\xi_i\xi_j \leq \nu'|\xi|^2 \quad \forall \xi \in I\!R^n \tag{44}$$

and some $\nu, \nu' > 0$. Then the strong solution of (40) verifies $\nabla u \in H_{loc}^{\gamma/2,\gamma}([0,T]\times\Omega)$ for some $\gamma \in]0,1[$ ([17], Ch. V., Thm. 3.1). It can be seen that the constant γ depends only on η, G and the constants C_1, C_2 in the definition of W_0. Thus the value of $\gamma \in]0,1[$ appearing in the definition of W_0 can be fixed once for all.

Now, let for each $w \in W_0$, $U(w)$ be the solution of (39) found above. We claim that choosing appropiately the constants C_1, C_2, C_3, C_4 and for some $T > 0$ sufficiently small $U(w) \in W_0$. Indeed we may take $C_1 = \|u^0\|_2 + 1$, $C_2 = \|\nabla u^0\|_\infty + 1$, $C_3 = \frac{1}{\eta} + \frac{2}{\eta^2}(\|\nabla u^0\|_\infty + 1)$, C_4 the constant given by ([17], Thm. 5.3). Then, if we choose T small enough, we see from the above estimates (42), (43) that $U(w) \in W_0$.

Finally, let us observe that U is weakly continuous from W_0 into W_0. Let $w_j \in W(0,T)$, $w_j \to w$ weakly in $W(0,T)$. Using the definition of $W(0,T)$ and the bound

(41) it is easy to see that $U(w_j) \to U(w)$ weakly in $W(0,T)$ (see [8]). Then, by Schauder's fixed point theorem there exists some $u \in W(0,T)$ such that $U(u) = u$. It follows that $u \in W_0$ is a solution of (38) in $[0,T] \times \Omega$ satisfying the estimates (41), (42), (43). Now let us prove the following estimates on $[0,T]$:

$$\|u(t)\|_2 + m \int_0^T \int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|) \leq \|u^0\|_2 \tag{45}$$

$$\int_0^T \int_\Omega |u_t|^2 + \int_\Omega \Theta(u(T))\varphi_\varepsilon(|\nabla u(T)|) \leq \int_\Omega \Theta(u^0)\varphi_\varepsilon(|\nabla u^0|) \tag{46}$$

To get the first estimate, multiply (38) by $\frac{1}{2}u$ and integrate by parts, to get:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega \Theta(u)\frac{|\nabla u|^2}{(\varepsilon^2 + |\nabla u|^2)^{1/2}} = \int_\Omega \Theta(u)^3 |\nabla G * u|^2 \varphi_\varepsilon(|\nabla u|) \leq \tag{47}$$

$$\leq \int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|) \leq \int_\Omega \Theta(u)\frac{|\nabla u|^2}{(\varepsilon^2 + |\nabla u|^2)^{1/2}}$$

and we get that $\|u(t)\|_2 \leq \|u^0\|_2$. With this the rigth hand side of (47) can be estimated by

$$\leq \frac{M^2}{\eta^2 + M^2}\int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|) \leq \frac{M^2}{\eta^2 + M^2}\int_\Omega \Theta(u)\frac{|\nabla u|^2}{(\varepsilon^2 + |\nabla u|^2)^{1/2}}$$

for some $M > 0$. Then (45) follows with $m = 1 - \frac{M^2}{\eta^2 + M^2}$. The second estimate (46) follows by multiplying (38) by u_t and integrating on Ω. Both estimates together imply that constants C, C' and C'' only depend on G and the initial datum $u_{0\varepsilon}$. Then solutions of (38) can be extended to $[0, +\infty[$ and both estimates (45), (46) also hold there. An L^∞ estimate will be also useful to pass to the limit as $\varepsilon \to 0$. For any $r \in [1, +\infty[$ write $u^r = |u|^r sign(u)$. By multiplying (38) by $\frac{1}{r}u^{r-1}$ and integrating by parts we get

$$\frac{1}{r}\frac{d}{dt}\int_\Omega |u|^r + (r-1)\int_\Omega |u|^{r-2}\Theta(u)\frac{|\nabla u|^2}{(\varepsilon^2 + |\nabla u|^2)^{1/2}} =$$

$$= \int_\Omega \Theta(u)^3 \varphi_\varepsilon(|\nabla u|)\nabla G * u \cdot \nabla G * u^{r-1} \leq$$

$$\leq \int_\Omega \Theta(u)^2 \varphi_\varepsilon(|\nabla u|)|\nabla G * u^{r-1}| \leq$$

$$\leq \frac{1}{\eta}\|\nabla G * u^{p-1}\|_\infty \int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|)$$

Since $\|\nabla G * u^{r-1}\|_\infty \leq \|\nabla G\|_r \||u|^{r-1}\|_{r'}$ (r' being the conjugate exponent of r) and $\||u|^{r-1}\|_{r'} = \|u\|_r$ the right hand side of above can be estimated by

$$\leq \frac{1}{\eta}\|\nabla G\|_r \|u\|_r^{r-1} \int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|)$$

Thus, we have

$$\frac{d}{dt}\|u(t)\|_r \leq \frac{1}{\eta}\|\nabla G\|_r \int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|)$$

It follows that

$$\|u(t)\|_r \leq \|u_{0\varepsilon}\|_r + \int_0^{+\infty} \int_\Omega \Theta(u)\varphi_\varepsilon(|\nabla u|) \tag{48}$$

which is finite in view of (45). In particular, letting $r \to \infty$ we get an L^∞ estimate for u.

2st step. Letting $\varepsilon \to 0$.

Let u_ε denote the strong solution of (37) (or (38)) found in the first step. Thus, letting $B_\varepsilon(t) = \partial_2 J^\varepsilon(u_\varepsilon(t), u_\varepsilon(t))$ which is given by the expression (31) we have that for all $v \in L^2(\Omega)$

$$J^\varepsilon(v, u_\varepsilon(t)) - J^\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) \geq - < u_{\varepsilon t} + B_\varepsilon(t), v - u_\varepsilon(t) > \tag{49}$$

From the above basic estimates (45), (46), (48) we may assume that, modulo a subsequence, there exists some function $u(t,x)$ in $L^\infty([0,+\infty[\times\Omega)$ with $u_t \in L^2([0,+\infty[, L^2(\Omega))$ $TV(u(t))$ bounded in $[0,+\infty[$ and such that

$$u_\varepsilon \to u \text{ in } L^r([0,T] \times \Omega) \text{ for all } r \in [1,+\infty[\text{ and a.e.} \tag{50}$$

$$\nabla G * u_\varepsilon \to \nabla G * u \text{ uniformly in } x, \text{ a.e. in } t \tag{51}$$

$$u_{\varepsilon t} \to u_t \text{ weakly in} L^2([0,T] \times \Omega) \tag{52}$$

Let $B(t) = \partial_2 J(u(t), u(t))$. Now let us check that if $v \in BV(\Omega)$

i) $J(v, u(t)) = \lim_{\varepsilon\to 0} J^\varepsilon(u_\varepsilon(t), u_\varepsilon(t))$

ii) $J(u(t), u(t)) \leq \liminf_{\varepsilon\to 0} J^\varepsilon(u_\varepsilon(t), u_\varepsilon(t))$

iii) $< u_{\varepsilon t}, v - u_\varepsilon(t) > \to < u_t, v - u(t) >$ weakly in $L^2([0,T]$

iv) $< B_\varepsilon(t), v - u_\varepsilon(t) > \to < B(t), v - u(t) >$

i) and *ii)* follow easily from (51) and the lower semicontinuity of the total variation with respect to weak* convergence. *iii)* follows directly from (50), (52). To prove *iv)* let us observe that

$$\int_\Omega |\nabla u_\varepsilon| \to \int_\Omega |\nabla u| \text{ weakly in } L^2(\Omega) \tag{53}$$

Indeed, since $\omega_\varepsilon := -div(\frac{\nabla u_\varepsilon}{(\varepsilon^2+|\nabla u_\varepsilon|^2)^{1/2}})$ is bounded in $L^2([0,T]\times\Omega)$ we may assume that $\omega_\varepsilon \to \omega$ weakly in $L^2([0,T]\times\Omega)$. Since $u_\varepsilon \to u$ strongly in $L^2([0,T]\times\Omega)$, then

$$\int_\Omega \omega\, u = \lim_{\varepsilon\to 0}\int_\Omega \omega_\varepsilon u_\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(\varepsilon^2+|\nabla u_\varepsilon|^2)^{1/2}} = \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon| \tag{54}$$

where the above convergences are taken weakly in $L^2([0,T])$. Now, write $TV_\varepsilon(v) := \int_\Omega(\varepsilon^2+|\nabla u_\varepsilon|^2)^{1/2}$. Then, for all $v\in BV(\Omega)$

$$TV_\varepsilon(v) - TV_\varepsilon(u_\varepsilon(t)) \geq -<\omega_\varepsilon, v-u_\varepsilon(t)>$$

and taking $\varepsilon\to 0$ we get that

$$TV(v) - TV(u(t)) \geq -<\omega, v-u> \text{ a.e. in } t$$

Choosing $v=0$ and $v=2u$ above we get that

$$\int_\Omega \omega\, u = TV(u)$$

This together with (54) proves (53). Now, (50) implies that $\nabla u_\varepsilon \to \nabla u$ weakly* as measures in $[0,T]\times\Omega$. Since we also have that

$$\int_0^T\int_\Omega |\nabla u_\varepsilon| \to \int_0^T\int_\Omega |\nabla u|\,, \tag{55}$$

using Lemma 2.1 in [11] it follows that

$$\int_0^T\int_\Omega f(t,x)\,|\nabla u_\varepsilon| \to \int_0^T\int_\Omega f(t,x)\,|\nabla u|\,, \tag{56}$$

for any continuous function f in $[0,T]\times\Omega$. Since $TV(u_\varepsilon(t))$ are uniformly bounded in t with abound independent of ε, it is easy to see that (56) holds for any $f\in L^2([0,T],C(\Omega))$ and

$$\int_0^T\int_\Omega f_\varepsilon(t,x)\,|\nabla u_\varepsilon| \to \int_0^T\int_\Omega f(t,x)\,|\nabla u|\,, \tag{57}$$

for any sequence $f_\varepsilon\to f$ in $L^2([0,T],C(\Omega))$, $f_\varepsilon, f\in L^2([0,T],C(\Omega))$. Since

$$<B_\varepsilon(t), u_\varepsilon(t)>= \int_\Omega \Theta(u_\varepsilon)^3\nabla G * u_\varepsilon\phi_\varepsilon(|\nabla u_\varepsilon|)\nabla G * u_\varepsilon$$

with a similar expression for $<B_\varepsilon(t), v>$, then $iv)$ is an immediate consequence of (57). Now letting $\varepsilon\to 0$ in (49) we get

$$J(v,u(t)) - J(u(t),u(t)) \geq -<u_t + B(t), v-u(t)> \text{ a.e. in } [0,T] \tag{58}$$

for all $v\in BV(\Omega)\cap L^2(\Omega)$, i.e., u is a strong solution of (33) in the sense of semigroups.

To see that J is a Lyapounov functional for (33), we consider first the approximate problems (38). Then it is straightforward to check that

$$\frac{d}{dt} J^{\varepsilon}(u_{\varepsilon}(t)) \leq 0$$

i.e., $J^{\varepsilon}(u_{\varepsilon}(t)) \leq J^{\varepsilon}(u_{\varepsilon}(s))$, for any $t, s \geq 0$, $t \geq s$. Now, since inequalities are maintained by weak limits, lettig $\varepsilon \to 0$ and using (51), (53) we get that $J(u(t)) \leq J(u(s))$ for any $t, s \geq 0$, $t \geq s$.

To study the assymptotic behavior as $t \to \infty$, we shall need the following estimate

$$\int_0^{\infty} \int_{\Omega} B(t,x)^2 < +\infty$$

whose proof is immediate by using (45). Since we also have $u \in L^{\infty}([0,\infty), L^2(\Omega)) \cap L^{\infty}([0,\infty), BV(\Omega))$ and $u_t \in L^2([0,\infty), L^2(\Omega))$, we may find a sequence $t_n \to \infty$ and a function $u \in BV(\Omega) \cap L^2(\Omega)$ such that $u(t_n) \to \bar{u}$, $u_t(t_n) \to 0$, $B(t_n) \to 0$ in $L^2(\Omega)$ as $t_n \to \infty$. Then, passing to the limit in (58) along the sequence t_n we find that

$$J(v, \bar{u}) \geq J(\bar{u}, \bar{u})$$

for all $v \in BV(\Omega) \cap L^2(\Omega)$. Taking $v = 0$, we have that $J(\bar{u}, \bar{u}) = 0$. It follows that $\bar{u}$ is a constant.□

Remark. We do not know if uniqueness hols for strong solutions of (32) or, in abstract version, (33). At the level of the approximate equation (38) or (37) with initial condition u^0 in $W^{1,\infty}(\Omega)$ there is uniqueness of strong solutions. Indeed this is not surprising due to the L^{∞} estimate on $|\nabla u|$. If we want to compare two solutions u_1 and u_2 with $|\nabla u_1|$, $|\nabla u_2|$ bounded by M, first we transform the term $div(\Theta(u)\,\frac{\nabla u}{(\varepsilon^2+|\nabla u|^2)^{1/2}})$ into a uniformly elliptic one of the form $div(\Theta(u)\,a(\nabla u))$ for some vector $a = (a_1, ..., a_n)$ with $a(p) = \frac{p}{(\varepsilon^2+|p|^2)^{1/2}}$ for $|p| \leq M$ and satisfying a uniform ellipticity condition like in (44). Now the proof becomes standard (see [8]). As in [18], we can get uniqueness for any initial condition in $W^{1,2}(\Omega)$ if we replace the term $div(\Theta(u)\,\frac{\nabla u}{(\varepsilon^2+|\nabla u|^2)^{1/2}})$ by a uniformly elliptic one of the form $div(\Theta(u)\,a(\nabla u))$ for some vector a satisfying a uniform ellipticity condition.

Remark. In this remark we adopt the notation of Section 2. To get a similar result for the constrained case, we define the regularized functional

$$F(u) = J^{\varepsilon}(u) + \beta \int_{\Omega} (Ku - u)^2 \tag{59}$$

where

$$J^{\varepsilon}(u) = \begin{cases} \displaystyle\int_{\Omega} \frac{\varphi_{\varepsilon}(|\nabla u|)}{(\eta^2 + |\nabla G * u|^2)^{1/2}} & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ \infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega) \end{cases} \tag{60}$$

and $\varphi_{\varepsilon}(|p|)$, $p \in I\!\!R^n$, is such that $a_{\varepsilon}(p) = \frac{\varphi'_{\varepsilon}(|p|)}{|p|}\, p$ satisfies the uniform ellipticity condition

$$\nu |\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial p_j}(p) \xi_i \xi_j \leq \nu' |\xi|^2 \quad \forall \xi \in I\!\!R^n \tag{61}$$

for some $\nu, \nu' > 0$. Then, we solve the partial differential equation

$$\begin{aligned} & u_t = div(\Theta(u)\, a_{\varepsilon}(\nabla u)) - Q\tilde{*}(\Theta(u)^3\, \nabla G * u\, \varphi_{\varepsilon}(|\nabla u|)) + \alpha(t) \\ & \quad -\lambda(t) A^*(Au - u_0) - \mu(I-K)^*(I-K)u \\ & \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \\ & \\ & u(0,x) = u^0(x) \text{ in } \Omega \end{aligned} \tag{62}$$

where $\alpha(t)$, $\lambda(t)$ given by

$$\alpha(t) = \int_{\Omega} Q\tilde{*}(\Theta(u)^3\, \nabla G * u\, \varphi_{\varepsilon}(|\nabla u|))$$

$$\lambda(t) = \frac{\displaystyle\int_{\Omega} A^*(Au - u_0)(\mathcal{A}(u) - \mu(I-K)^*(I-K)u)}{\|A^*(Au - u_0)\|_2}$$

with

$$\mathcal{A}(u) = div(\Theta(u)\, a_{\varepsilon}(\nabla u)) - Q\tilde{*}(\Theta(u)^3\, \nabla G * u\, \varphi_{\varepsilon}(|\nabla u|))$$

are the terms that help to keep the constraints $\int_{\Omega} Au = \int_{\Omega} u_0$ and $\int_{\Omega} |Au - u_0|^2 = \sigma^2$, respectively.

Indeed, with arguments similar to the previous ones and to the ones in [18] one proves that if A, K satisfy the assumptions $H1 - H5$ above with $p = 2$, $R(A)$ (the range of A) is dense in $L^2(\Omega)$, A, A^* are bounded operators in $W^{1,2}(\Omega)$ and $u_0, u^0 \in W^{1,2}(\Omega)$ are such that $\int_{\Omega} |Au^0 - u_0|^2 = \sigma^2$ then (62) has a unique solution $u \in C([0,\infty[, L^2(\Omega)) \cap L^{\infty}([0,\infty[, W^{1,2}(\Omega))$, $u_t \in L^2([0,\infty[, L^2(\Omega))$ and $\lambda \in L^{\infty}([0,T])$ for all $T \in (0,\infty)$. This solution satisfies the constraints above. Moreover, for any sequence $t_k \to \infty$, there exists a subsequence t_{k_j} such that $u(t_{k_j})$ converges to $\bar{u}$ satisfying

$$\partial_1 J(\bar{u}, \bar{u}) + \partial_2 J(\bar{u}, \bar{u}) - \bar{\alpha} \ni -\bar{\lambda} A^*(Au - u_0) - \beta(I-K)^*(I-K)\bar{u}$$

for some $\bar{\alpha}, \bar{\lambda} \in I\!\!R$.

References

[1] R. Acar and C. R. Vogel, *Analysis of Total Variation Penalty Methods for Ill-Posed Problems*, Inverse Problems, 10 (1994), pp. 1217-1229.

[2] L. Alvarez, F. Guichard, P. L. Lions, and J. M. Morel, *Axioms and fundamental equations of image processing*, Arch. Rational Mechanics and Anal. , 16, IX (1993), pp. 200-257.

[3] L. Alvarez, P. L. Lions, and J. M. Morel, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM J. Numer. Anal. 29 (1992) pp. 845-866.

[4] H. Attouch and A. Damlamian, *Problemes d'evolution dans les Hilbert et applications*, J. Math. Pures et Appl. 54 (1975) pp. 53-74.

[5] J. P. Aubin, *Optima and Equilibria*, GTM 140, Springer Verlag, 1993.

[6] H. Brezis, *Operateurs Maximaux Monotones*, North Holland, Amsterdam, 1973.

[7] V. Caselles, B. Coll and G. Sapiro, *PDE's for Image Segmentation*, In preparation.

[8] F. Catte, P. L. Lions, J. M. Morel and B. Coll, *Image Selective Smoothing and Edge Detection by Nonlinear Diffusion*, SIAM J. Numer. Anal. .

[9] A. Chambolle and P. L. Lions, *Image Recovery via Total Variation Minimization and Related Problems*, Preprint, 1995.

[10] P. Charbonnier, L. Blanc-Feraud and M. Barlaud, *An Adaptative Reconstruction Method involving Discontinuities*, Tech. Rep. TR# 92-61, I3S, CNRS URA 1376, Sophia-Antipolis, France, 1992.

[11] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North Holland, Amsterdam, 1976.

[12] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992.

[13] L. C. Evans, *Convergence of an Algorithm for Mean Curvature Motion*, Preprint, 1992.

[14] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.

[15] D. Geman and G. Reynolds, *Constrained Image Restoration and the Recovery of Discintinuities*, PAMI, 14 (1992), pp. 367-383.

[16] B.R. Hunt, *The Application of Constrained Least Squares Estimation to Image Restoration by Digital Computer*, IEEE Trans. Comp., Vol. C-22 (1973), pp. 805-812.

[17] O. A. Ladyzhenskaja, V. A. Solonnikov and N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Trans. Math. Monographs 23, American Math. Society, Rhode Island, 1968.

[18] P. L. Lions, S. Osher and L. Rudin, *Denoising and Deblurring using Constrained Nonlinear Partial Differential Equations*, Tech. Repport, Cognitech Inc., Santa Monica, CA, 1992, submitted to SINUM.

[19] D. L. Phillips, *A Technique for the Numerical Solution of Certain Integral Equations of the First Kind*, J. ACM, 9 (1962), pp. 84-97.

[20] J. G. Rosen, *The Gradient Projection Method for Nonlinear Programming. Part II, Nonlinear Constraints*, J. Soc. Indust. Appl. Math., 9 (1961), pp. 514-532.

[21] L. Rudin, *Segmentation and Restoration using Local Constraints*, Technical Repport 36, Cognitech Inc., Santa Monica, CA, 1993.

[22] L. Rudin and S. Osher, *Total Variation based Image Restoration with Free Local Constraints*, Proc. of the IEEE ICIP-94, vol. 1, Austin, TX, 1994, pp. 31-35.

[23] L. Rudin, S. Osher and E. Fatemi, *Nonlinear Total Variation based Noise Removal Algorithms*, Physica D., 60 (1992), pp. 259-268.

[24] G. Sapiro and A. Tannenbaum, *On Affine Plane Curve Evolution*, Journal of Functional Analysis, 119:1 (1994), pp. 79-120.

[25] G. Sapiro and A. Tannenbaum, *Affine Invariant Scale-Space*, Int. Journal of Computer Vision, 11:1 (1993) pp. 25-44.

[26] R. Temam, *Problemes Mathematiques en Plasticite*, Methodes Mathematiques de l'Informatique, Gauthier-Villars, 1983.

[27] S. Twomey, *On the Numerical Solution of Fredholm Integral Equations of the First Kind by the Inversion of the Linear System Produced by Quadrature*, J. ACM, 10 (1963), pp. 97-107.

[28] S. Twomey, *The Application of Numerical Filtering to the Solution of Integral Equations Encountered in Indirect Sensing Measurements*, J. Franklin Inst., 297 (1965), pp. 95-109.

[29] C. R. Vogel and M. E. Oman, *Iterative Methods for Total Variation Denoising*, SIAM J. Sci. Computing, to appear.

[30] W. P. Ziemer, *Weakly Differentiable Functions*, GTM 120, Springer Verlag, 1989.